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Abstract. In the jet-bundle description of first-order classical field theories there are some elements, such as the Lagrangian energy and the construction of the Hamiltonian formalism, which require the prior choice of a connection. Bearing these facts in mind, we analyse the situation in the jet-bundle description of time-dependent classical mechanics. We prove that this connection-dependence also occurs in this case, although it is usually hidden by the use of the 'natural' connection given by the trivial bundle structure of the phase spaces under consideration. However, we also prove that this dependence is dynamically irrelevant, except where the dynamical variation of the energy is concerned. In addition, the relationship between first integrals and connections is shown for a sufficiently large class of Lagrangians.

1. Introduction

One of the most interesting lines of current research in mathematical physics is the geometric formulation of first-order classical field theories, which is mainly achieved by the use of jet bundles $J^1E \to E \to M$ and the geometrical structures with which they are endowed [2, 3, 6, 8–12, 14–16, 18].

Among all the relevant features observed when dealing with these geometrical formulations, we wish to point out the following: there are some dynamical elements of the theories depending on the prior choice of a connection in the configuration bundle $\pi\colon E \to M$. For instance:

- In the Lagrangian formalism, the definition of the *density of Lagrangian energy* and the *Lagrangian energy function* [8].
- The construction of the Hamiltonian formalism of these theories, in particular the *Hamiltonian function* and the Liouville form. (see, for instance, [3]).

It is also known that time-dependent mechanical systems can be geometrically described using jet bundles. Then $M = \mathbb{R}$, $E = Q \times \mathbb{R}$ and $J^1E = TQ \times \mathbb{R}$ (where Q represents the configuration space of the system), and consequently this can be considered as a particular situation of field theories [7, 9]. In this context, since $Q \times \mathbb{R}$ is a trivial bundle, it is canonically endowed with a 'natural' connection which is used (when necessary) to define all the dynamical and geometric elements of the theory; in particular, the Hamiltonian formalism and the energy Lagrangian function. A possible conclusion of this approach is

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that, for non-autonomous mechanical systems, the definition of these dynamical elements does not depend on the choice of any connection.

The aim of this paper is to make evident the influence of the choice of a connection in the geometrical construction of some elements of the theory. In order to achieve this, we will choose an arbitrary connection in the bundle $Q \times \mathbb{R} \to \mathbb{R}$ and we will reconstruct the dynamics of the theory starting from this point. In addition, we will study the relation among the descriptions coming from different choices of a connection, and we will interpret the results of this analysis from a dynamical point of view. Connections on $Q \times \mathbb{R} \to \mathbb{R}$ can be understood as special time-actions in the manifold $Q \times \mathbb{R}$. The standard one is by translations $(q, t) \overset{s}{\mapsto} (q, t + s)$. Other actions correspond to non-standard connections. In this sense changes in connections imply changes in the energy and in the geometric elements of the theory, but there are no changes in the dynamical evolution of the system if the new geometric elements and the Hamiltonian deformed by the connection are taken.

The structure of the work is as follows. In section 2 we introduce the basic ideas about connections in the bundle $Q \times \mathbb{R}$ and the natural geometric elements in the bundle $TQ \times \mathbb{R} \to Q \times \mathbb{R}$. Section 3 is devoted to presenting the Lagrangian formalism (using arbitrary connections) and showing which of its elements are connection-depending. We obtain results on the variation of the energy along the motion of the system and on the relationship between first integrals and connections. All these results generalize classical ones for non-autonomous systems. In section 4 we construct the Hamiltonian formalism for non-autonomous systems, depending on the choice of an arbitrary connection and, subsequently, we give a dynamical interpretation of the results so obtained. In section 5 a characterization of the Lagrangian energy function, based on variational principles, is given. Section 6 is devoted to summarizing the conclusions reached in the work.

All the manifolds and maps are C^∞. Summation over repeated indices is understood.

2. The 1-jet bundle of $\pi: Q \times \mathbb{R} \to \mathbb{R}$. Geometric structures and connections

The ideas in this section are known. We merely emphasize the differences between the general situation and this particular one in order to make the paper more readable and self-contained. See [13] and [19] as general references.

2.1. Connections in $\pi : Q \times \mathbb{R} \to \mathbb{R}$

Consider the bundle $\pi: Q \times \mathbb{R} \to \mathbb{R}$, where Q is an n-dimensional differentiable manifold (the *configuration space* of a physical system). The 1-jet bundle of sections of π is $\pi_1: TQ \times \mathbb{R} \to Q \times \mathbb{R}$. In fact, if $\phi = (\phi_Q, \mathrm{Id})$ is a local section of π defined in a neighbourhood of $s \in \mathbb{R}$, with $\phi(s) = (q, s)$, then the 1-jet equivalence class of ϕ is determined by $(T_s \phi_Q)(\mathrm{d}/\mathrm{d}t)$; that is, an element of $T_q Q$. Conversely, if $v \in T_q Q$ and $s \in \mathbb{R}$, there is an equivalence class of curves $\phi_Q: (s - \epsilon, s + \epsilon) \to Q$ with $T_s \phi_Q (\mathrm{d}/\mathrm{d}t) = v$; so a 1-jet equivalence class of local sections is defined and $\phi = (\phi_Q, \mathrm{Id})$ is one of its representatives.

If $(U; q^\mu)$ is a local chart in Q, then a local chart in $TQ \times \mathbb{R}$ is $(\pi_1^{-1}(U \times \mathbb{R}); q^\mu, t, v^\mu)$ where

$$v^\mu((q, s), v) = \left(\frac{\mathrm{d}\phi^\mu(t)}{\mathrm{d}t} \right)_s$$

$\phi: \mathbb{R} \to Q \times \mathbb{R}$ being a representative of $((q, s), v) \in TQ \times \mathbb{R}$ and $\phi^\mu = q^\mu \circ \phi_Q$. Of course, these coordinates v^μ are the physical velocities.

In order to introduce the connections, we must study the tangent bundle of $Q \times \mathbb{R}$. Observe that there is a natural identification between $T(Q \times \mathbb{R})$ and $TQ \times T\mathbb{R} := \pi_Q^* TQ \oplus \pi^* T\mathbb{R}$ given by

$$\psi \quad : \quad T(Q \times \mathbb{R}) \quad \longrightarrow \quad TQ \times T\mathbb{R}$$
$$((q, s), u) \quad \mapsto \quad ((q, s), T_{(q,s)} \pi_Q(u) + T_{(q,s)} \pi(u))$$

where $\pi_Q: Q \times \mathbb{R} \to Q$ is the natural projection.

But $\pi_Q^* TQ$ is identified with $V(\pi)$ (the vertical sub-bundle of $T(Q \times \mathbb{R})$ with respect to π). In fact, if $(q_o, s) \in Q \times \mathbb{R}$ and $j_s: Q \to Q \times \mathbb{R}$ is the s-injection defined by $j_s(q) = (q, s)$, then $V_{(q_o, s)}(\pi) = T_{q_o} j_s(T_{q_o} Q)$. So we have the natural splitting

$$T(Q \times \mathbb{R}) = V(\pi) \oplus \pi^* T\mathbb{R}$$

and $\pi^* T\mathbb{R}$ is called the *horizontal sub-bundle*. As a consequence, if $v \in T_{(q,s)}(Q \times \mathbb{R})$, we will write $v = v_Q + v_{\mathbb{R}}$ in this splitting.

This natural splitting will be called the *standard connection* in the bundle $\pi: Q \times \mathbb{R} \to \mathbb{R}$. The theory of connections describes the possible splittings of this kind.

Following this model we have that:

Proposition 1. The following elements on $\pi: Q \times \mathbb{R} \to \mathbb{R}$ can be canonically constructed one from the other:

(i) A section of $\pi_1: TQ \times \mathbb{R} \to Q \times \mathbb{R}$, that is a mapping $\nabla: Q \times \mathbb{R} \to TQ \times \mathbb{R}$ such that $\pi_1 \circ \nabla = \mathrm{Id}_{Q \times \mathbb{R}}$.

(ii) A sub-bundle $H(\nabla)$ of $T(Q \times \mathbb{R})$ such that

$$T(Q \times \mathbb{R}) = V(\pi) \oplus H(\nabla). \tag{1}$$

(iii) A semibasic 1-form $\tilde{\nabla}$ on $Q \times \mathbb{R}$ with values in $T(Q \times \mathbb{R})$ (that is, an element of $\Gamma(Q \times \mathbb{R}, \pi^* T^* Q) \otimes \mathcal{X}(Q \times \mathbb{R}))$, such that $\alpha \circ \tilde{\nabla} = \alpha$, for every semibasic form $\alpha \in \Omega^1(Q \times \mathbb{R})$.

(We use the notation $\Gamma(A, B)$ for the set of sections of the bundle $B \to A$).

Proof. The proof of this statement, in the general case of a bundle $\pi: E \to M$, can be found in (different sections of) [19]. $\qquad\square$

Definition 1. A *connection* in the bundle $\pi: Q \times \mathbb{R} \to \mathbb{R}$ is one of the above mentioned equivalent elements. $H(\nabla)$ is called the *horizontal sub-bundle* of $T(Q \times \mathbb{R})$ associated with the connection ∇ and its sections *horizontal vector fields*. $\tilde{\nabla}$ is called the *connection form*.

Given the sub-bundle $H(\nabla)$ and the splitting (1), we have the maps

$$h_\nabla: T(Q \times \mathbb{R}) \to H(\nabla) \qquad v_\nabla: T(Q \times \mathbb{R}) \to V(\pi)$$

called the *horizontal* and *vertical projections* (we will use the same symbols h_∇ and v_∇ for the natural extensions of these maps to vector fields).

In a local chart (q^μ, t, v^μ) the expressions of all these elements are

$$\nabla(q, s) = (q, s, \gamma^\mu(q, s))$$

$$\tilde{\nabla} = \mathrm{d}t \otimes \left(\frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} \right)$$

$$H(\nabla) = \mathrm{span} \left\{ \frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} \right\}$$

(for every $(q, s) \in Q \times \mathbb{R}$). The relations among all of them are given locally by their 'coordinates' $\gamma^\mu(q, t)$.

For every vector field

$$X \equiv X_\mathbb{R} + X_Q \equiv f \frac{\partial}{\partial t} + X_Q = f \frac{\partial}{\partial t} + \lambda^\mu \frac{\partial}{\partial q^\mu} \in \mathcal{X}(Q \times \mathbb{R})$$

where $X_Q \in C^\infty(Q \times \mathbb{R}) \otimes \mathcal{X}(Q)$, the horizontal and vertical projections are given by

$$X_{h_\nabla} \equiv h_\nabla(X) = f \left(\frac{\partial}{\partial t} + \lambda^\mu \frac{\partial}{\partial q^\mu} \right) \qquad X_{v_\nabla} \equiv v_\nabla(X) = X_Q - f \lambda^\mu \frac{\partial}{\partial q^\mu}$$

and we have the splitting

$$X = f \frac{\partial}{\partial t} + X_Q = X - \tilde{\nabla}(X) + \bar{\nabla}(X) = X_{v_\nabla} + X_{h_\nabla}.$$

Moreover, we have the following result.

Proposition 2.   Every connection in the bundle $\pi : Q \times \mathbb{R} \to \mathbb{R}$ induces a canonical lifting $\mathcal{X}(\mathbb{R}) \longrightarrow \mathcal{X}(Q \times \mathbb{R})$. If $\overline{X}$ is the image of $X \in \mathcal{X}(\mathbb{R})$ by this lifting, then $\overline{X}$ is a horizontal vector field.

Proof.   Let ∇ be a connection, $(q, s) \in Q \times \mathbb{R}$ and and $\phi = (\phi_Q, \text{Id})$ a representative of $\nabla(q, s)$. If $X \in \mathcal{X}(\mathbb{R})$, we define $\overline{X}$ by

$$\overline{X}(q, s) := T_s \phi(X_s).$$

From the local expression of ∇ we deduce that $\overline{X}$ is a C^∞-vector field and it is horizontal because

$$\tilde{\nabla}((q, s), \overline{X}(q, s)) = (T_s \phi \circ T_{(q,s)} \pi)(\overline{X}_{(q,s)}) = (T_s \phi \circ T_{(q,s)} \pi \circ T_s \phi)(X_s)$$
$$= T_s \phi(X_s) = \overline{X}_{(q,s)}.$$

$\square$

But $\mathcal{X}(\mathbb{R})$ has a global generator $\mathrm{d}/\mathrm{d}t$; so, given a connection ∇, we can take its lifting:

$$\left. \frac{\overline{\mathrm{d}}}{\mathrm{d}t} \right|_{(q,s)} = T_s \phi_Q \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_s \right) + T_s \text{Id}_q \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_s \right) = T_s \phi_Q \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_s \right) + \left. \frac{\partial}{\partial t} \right|_{(q,s)}.$$

Observe that the map

$$(q, s) \mapsto T_s \phi_Q \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_s \right)$$

is a section of the bundle $\pi^* TQ \to Q \times \mathbb{R}$, that is, a time-dependent vector field in Q. Then we have:

Proposition 3.   A connection in the bundle $\pi : Q \times \mathbb{R} \to \mathbb{R}$ is equivalent to a time-dependent vector field in Q.

Proof.   Given a connection ∇, taking

$$Y = \frac{\overline{\mathrm{d}}}{\mathrm{d}t} - \frac{\partial}{\partial t}$$

we have the desired vector field.

Conversely, given a time-dependent vector field $Y: Q \times \mathbb{R} \to \pi_Q^* TQ$, we have a connection defined by

$$\nabla \ : \ \begin{array}{ccc} Q \times \mathbb{R} & \longrightarrow & TQ \times \mathbb{R} \\ (q, s) & \mapsto & (Y_{(q,s)}, s) \end{array}$$

$\square$

If Y is the vector field induced by ∇, then the connection form is written as

$$\tilde{\nabla} = dt \otimes \left(\frac{\partial}{\partial t} + Y \right).$$

As we have pointed out above, the trivial bundle $\pi: Q \times \mathbb{R} \to \mathbb{R}$ has a natural connection: the standard one ∇_0, with $H(\nabla_0) = \pi^* T\mathbb{R}$. In this case

$$\tilde{\nabla}_0 = dt \otimes \frac{\partial}{\partial t}$$

the time-dependent vector field associated with ∇_0 is $Y_0 = 0$ and the lifting induced by ∇_0 is given by

$$\left. \frac{d}{dt} \right|_{(q,s)} = \left. \frac{\partial}{\partial t} \right|_{(q,s)}.$$

If we have another connection ∇ with associated vector field Y, this lifting is

$$\left. \frac{d}{dt} \right|_{(q,s)} = \left. \frac{\partial}{\partial t} \right|_{(q,s)} + Y_{(q,s)}$$

then we can understand that the 'lines of time' induced by this connection are the integral curves of the vector field

$$\frac{\partial}{\partial t} + Y := \bar{Y}$$

which is called the *suspension* of Y [1].

From now on, we will refer to *non-standard connections* in the bundle $\pi: Q \times \mathbb{R}$ in those cases that differ from the standard one.

2.2. *Geometric elements*

In the Lagrangian formalism the dynamics takes place in the manifold $TQ \times \mathbb{R}$. Then, in order to set it, we need to introduce some geometrical elements of the bundle $\pi_1 TQ \times \mathbb{R} \to Q \times \mathbb{R}$ (see [7, 9, 11, 19] for details). When needed, we will use a local system given by (q^μ, t, v^μ).

2.2.1. *The structural canonical 1-form [9].* We can define a 1-form ϑ in $TQ \times \mathbb{R}$, with values in $\pi_1^* V(\pi)$, in the following way:

$$\vartheta(((q, s), u); X) := (T_{((q,s),u)} \pi_1 - T_{((q,s),u)}(\phi \circ \pi \circ \pi_1))(X_{(q,s)})$$

where ϕ is a representative of $((q, s), u) \in TQ \times \mathbb{R}$. ϑ is called the *structure canonical form* of $TQ \times \mathbb{R}$. Its local expression is

$$\vartheta = (dq^\mu - v^\mu dt) \otimes \frac{\partial}{\partial q^\mu}.$$

2.2.2. *The vertical endomorphisms.* Taking into account that $\pi_1 : TQ \times \mathbb{R} \to Q \times \mathbb{R}$ is a vector bundle and the fibre on $(q, s) \in Q \times \mathbb{R}$ is $T_q Q \times \{s\}$, there exists a canonical diffeomorphism between the π_1-vertical sub-bundle and $\pi_1^*(TQ \times \mathbb{R})$, that is

$$V(\pi_1) \simeq \pi_1^*(TQ \times \mathbb{R}) \simeq \pi_1^* \pi_Q^* TQ \simeq \pi_1^* V(\pi).$$

We denote by $S : \pi_1^* V(\pi) \to V(\pi_1)$ the realization of this isomorphism and we will use the same notation S for its action on the modules of sections of these bundles.

By construction we have that, locally

$$S \left(\frac{\partial}{\partial q^\mu} \right) = \frac{\partial}{\partial v^\mu}$$

and so

$$S = \xi^\mu \otimes \frac{\partial}{\partial v^\mu}$$

where $\{\xi^\mu\}$ is the dual basis of $\{\partial/\partial q^\mu\}$ in $\pi_1^* V(\pi)$.

Here S is an element of $\Gamma(TQ \times \mathbb{R}, \pi_1^* V^*(\pi)) \otimes \Gamma(TQ \times \mathbb{R}, V(\pi_1))$. Taking into account that the structure form ϑ is an element of $\Omega^1(TQ \times \mathbb{R}, \pi_1^* V(\pi_1)) = \Omega^1(TQ \times \mathbb{R}) \otimes \Gamma(TQ \times \mathbb{R}, \pi_1^* V(\pi))$, using the natural duality, by contracting S with ϑ we obtain an element

$$\mathcal{V} := i(S)\vartheta \in \Omega^1(TQ \times \mathbb{R}) \otimes \Gamma(TQ \times \mathbb{R}, V(\pi_1))$$

whose local expression is

$$\mathcal{V} = (dq^\mu - v^\mu dt) \otimes \frac{\partial}{\partial v^\mu}.$$

Notice that $\mathcal{V}$ can be thought as a $C^\infty(TQ \times \mathbb{R})$-module morphism $\mathcal{V} : \mathcal{X}(TQ \times \mathbb{R}) \to \mathcal{X}(TQ \times \mathbb{R})$ with image on the π_1-vertical vector fields.

S and $\mathcal{V}$ are called the *vertical endomorphisms* of $TQ \times \mathbb{R}$. They are sections of different bundles but, if we have a connection ∇, S can be also understood as an endomorphism on $\mathcal{X}(TQ \times \mathbb{R})$. In fact, notice that the splitting $T(Q \times \mathbb{R}) = V(\pi) \oplus H(\nabla)$, induced by the connection ∇, has a dual counterpart $T^*(Q \times \mathbb{R}) = V^*(\pi) \oplus H^*(\nabla)$. So $V^*(\pi)$ is identified by ∇ with a sub-bundle of $T^*(Q \times \mathbb{R})$. With the same notation as above, if $\{\xi^\mu\}$ is the dual basis of $\{\partial/\partial q^\mu\}$ in $\pi_1^* V(\pi)$, then this identification is given by

$$\xi^\mu \mapsto dq^\mu - \gamma^\mu dt$$

if

$$\nabla = dt \otimes \left(\frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} \right).$$

In the same way, we have the splitting $\pi_1^* T^*(Q \times \mathbb{R}) = \pi_1^* V^*(\pi) \oplus \pi_1^* H^*(\nabla)$ and the injection $j_\nabla : \pi_1^* V^*(\pi) \hookrightarrow \pi_1^* T^*(Q \times \mathbb{R})$. But this last bundle is a sub-bundle of $T^*(TQ \times \mathbb{R})$ whose sections are the π_1-semibasic 1-forms in $TQ \times \mathbb{R}$. Hence, by means of this injection, S is an element of $\Omega^1(TQ \times \mathbb{R}) \otimes \mathcal{X}(TQ \times \mathbb{R})$, with values in the vertical vector fields, which will be denoted by S^∇. Its local expression is

$$S^\nabla = (dq^\mu - \gamma^\mu dt) \otimes \frac{\partial}{\partial v^\mu}$$

Now, we are able to consider the difference $S^\nabla - \mathcal{V} \in \Omega^1(TQ \times \mathbb{R}) \otimes \mathcal{X}(TQ \times \mathbb{R})$ whose local expression is

$$S^\nabla - \mathcal{V} = (v^\mu - \gamma^\mu) dt \otimes \frac{\partial}{\partial t}$$

which will henceforth be used in order to characterize the Lagrangian energy.

3. Lagrangian formalism. Connections and Lagrangian energy functions

A *time-dependent Lagrangian function* is a function $\mathcal{L} \in C^\infty(TQ \times \mathbb{R})$. As is known, we can construct the Lagrangian forms associated with $\mathcal{L}$ using the geometrical structure of $TQ \times \mathbb{R}$.

Definition 2.   The *Poincaré–Cartan 1- and 2-forms* associated with the Lagrangian function $\mathcal{L}$ are the forms in $TQ \times \mathbb{R}$ defined by

$$\Theta_{\mathcal{L}} := d\mathcal{L} \circ \mathcal{V} + \mathcal{L}dt \qquad \Omega_{\mathcal{L}} := -d\Theta_{\mathcal{L}}.$$

For the coordinate expressions of the Poincaré–Cartan forms we obtain

$$\Theta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial v^\mu}(dq^\mu - v^\mu dt) + \mathcal{L}dt = \left(\mathcal{L} - v^\mu \frac{\partial \mathcal{L}}{\partial v^\mu}\right)dt + \frac{\partial \mathcal{L}}{\partial v^\mu}dq^\mu$$

$$\Omega_{\mathcal{L}} = -d\left(\frac{\partial \mathcal{L}}{\partial v^\mu}\right) \wedge dq^\mu + d\left(\frac{\partial \mathcal{L}}{\partial v^\mu}v^\mu - \mathcal{L}\right) \wedge dt.$$

Observe that these elements do not depend on the connection.

As usual, we say that a Lagrangian function $\mathcal{L}$ is *regular* iff its associated form $\Omega_{\mathcal{L}}$ has maximal rank, which is equivalent to demanding that $\det\left(\partial^2 \mathcal{L}/\partial v^\mu \partial v^\nu\right)$ is different from zero at every point.

Assuming the regularity of $\mathcal{L}$, the dynamics of the system is described by a vector field $X_{\mathcal{L}} \in \mathcal{X}(TQ \times \mathbb{R})$, which is a *second-order differential equation* (SODE), such that

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0 \qquad i(X_{\mathcal{L}})dt = 1 \tag{2}$$

As a consequence, the integral curves of $X_{\mathcal{L}}$ verify the *Euler–Lagrange equations* and $X_{\mathcal{L}}$ is obviously independent of any connection you can choose on $Q \times \mathbb{R} \to \mathbb{R}$.

A very different picture arises when we try to define intrinsically the *Lagrangian energy function*. If we consider the standard connection, it can be obtained as follows: take the lifting of d/dt from $\mathbb{R}$ to $TQ \times \mathbb{R}$ given by the connection, which will be denoted as usual by $\partial/\partial t$, then

$$E_{\mathcal{L}} = -i\left(\frac{\partial}{\partial t}\right)\Theta_{\mathcal{L}}$$

its local expression being

$$E_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial v^\mu}v^\mu - \mathcal{L}.$$

One of the most significant aspects of the above expression for the Lagrangian energy is that it is obtained by contraction of the infinitesimal time-action generator with the Lagrangian form. If we consider a non-standard connection ∇, then in order to define the Lagrangian energy function associated to ∇ we must lift d/dt from $\mathbb{R}$ to $TQ \times \mathbb{R}$ to be contracted with $\Theta_{\mathcal{L}}$ in the following way:

Definition 3.   Let ∇ be a connection in $\pi \colon Q \times \mathbb{R} \to \mathbb{R}$, $\tilde{Y} \in \mathcal{X}(Q \times \mathbb{R})$ the lifting of d/dt induced by ∇ and $j^1\tilde{Y} \in \mathcal{X}(TQ \times \mathbb{R})$ its canonical lifting. The *Lagrangian energy function* associated with the Lagrangian function $\mathcal{L}$ and the connection ∇ is

$$E_{\mathcal{L}}^{\nabla} = -i(j^1\tilde{Y})\Theta_{\mathcal{L}}$$

In a local chart, if

$$\tilde{Y} = \frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu}$$

we have

$$j^1\tilde{Y} = \frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} + \left(\frac{\partial \gamma^\mu}{\partial t} + v^\nu \frac{\partial \gamma^\mu}{\partial q^\nu}\right)\frac{\partial}{\partial v^\mu}$$

and

$$E_{\mathcal{L}}^\nabla = \frac{\partial \mathcal{L}}{\partial v^\mu}(v^\mu - \gamma^\mu) - \mathcal{L}.$$

It is obvious from this expression that the Lagrangian energy is connection-depending. In particular, when the standard connection is used, then $\gamma^\mu = 0$, which is the fact that 'hides' the explicit dependence on the connection of the energy in classical mechanics.

Notice that, since $\Theta_{\mathcal{L}}$ is π_1-semibasic, the energy function does not, in fact, depend on the extension of $\tilde{Y}$ from $Q \times \mathbb{R}$ to $TQ \times \mathbb{R}$.

Another characterization of the energy can be obtained in the following way. In the geometrical description of mechanics, it is also usual to define the Lagrangian energy using the vertical endomorphism. In order to achieve this result, consider the difference $S^\nabla - \mathcal{V} \in \Omega^1(TQ \times \mathbb{R}) \otimes \Gamma(TQ \times \mathbb{R}, V(\pi_1))$ and its natural contraction with $d\mathcal{L}$, $d\mathcal{L} \circ (S^\nabla - \mathcal{V}) \in \Omega^1(TQ \times \mathbb{R})$, whose local expression is

$$d\mathcal{L} \circ (S - \mathcal{V}) = \frac{\partial \mathcal{L}}{\partial v^\mu}(v^\mu - \gamma^\mu)dt.$$

Then it is easy to verify that $E_{\mathcal{L}}^\nabla = i(j^1\tilde{Y})(d\mathcal{L} \circ (S - \mathcal{V}) - \mathcal{L}dt)$.

The 1-form $\mathcal{E}_{\mathcal{L}}^\nabla := d\mathcal{L} \circ (S - \mathcal{V}) - \mathcal{L}dt \in \Omega^1(TQ \times \mathbb{R})$ is called the *density of Lagrangian energy* associated with the Lagrangian function $\mathcal{L}$ and the connection ∇ and it is a $(\pi \circ \pi_1)$-semibasic form whose local expressions is

$$\mathcal{E}_{\mathcal{L}}^\nabla = \left(\frac{\partial \mathcal{L}}{\partial v^\mu}(v^\mu - \gamma^\mu) - \mathcal{L}\right)dt.$$

At this point it is relevant to ask about the dynamical meaning of these connection-depending energies. Remember that in the standard case we have

$$X_{\mathcal{L}}(E_{\mathcal{L}}) = -\frac{\partial \mathcal{L}}{\partial t} \tag{3}$$

which is a relation between the rates of variation of two functions with respect to different time actions: the action through the flow of the dynamical vector field $X_{\mathcal{L}}$ and the one given by translations in time. As we have seen above, every connection ∇ induces a lifting $j^1\tilde{Y}$ of d/dt, which is the infinitesimal generator of a new time action, and then the equivalent result to (3) is given by the following:

Theorem 1. Let $X_{\mathcal{L}} \in \mathcal{X}(TQ \times \mathbb{R})$ be the dynamical vector field (the solution of equations (2)). Then

$$X_{\mathcal{L}}(E_{\mathcal{L}}^\nabla) = -(j^1\tilde{Y})\mathcal{L}.$$

Proof. Notice that, since $X_{\mathcal{L}}$ is a SODE, $\mathcal{V}(X_{\mathcal{L}}) = 0$; therefore

$$X_{\mathcal{L}}(E_{\mathcal{L}}^{\nabla}) = i(X_{\mathcal{L}})\mathrm{d}E_{\mathcal{L}}^{\nabla} = -i(X_{\mathcal{L}})\mathrm{d}i(j^1\tilde{Y})\Theta_{\mathcal{L}} = i(X_{\mathcal{L}})i(j^1\tilde{Y})\mathrm{d}\Theta_{\mathcal{L}} - i(X_{\mathcal{L}})\mathrm{L}(j^1\tilde{Y})\Theta_{\mathcal{L}}$$

$$= i(j^1\tilde{Y})i(X_{\mathcal{L}})\Omega_{\mathcal{L}} + i([j^1\tilde{Y}, X_{\mathcal{L}}])\Theta_{\mathcal{L}} - \mathrm{L}(j^1\tilde{Y})i(X_{\mathcal{L}})\Theta_{\mathcal{L}}$$

$$= -\mathrm{L}(j^1\tilde{Y})i(X_{\mathcal{L}})(\mathrm{d}\mathcal{L}\circ\mathcal{V} + \mathcal{L}\mathrm{d}t)$$

$$= -\mathrm{L}(j^1\tilde{Y})i(\mathcal{V}(X_{\mathcal{L}}))\mathrm{d}\mathcal{L} - \mathrm{L}(j^1\tilde{Y})(\mathcal{L}i(X_{\mathcal{L}})\mathrm{d}t)$$

$$= -(j^1\tilde{Y})\mathcal{L}$$

where we have taken into account that $[j^1\tilde{Y}, X_{\mathcal{L}}]$ is π_1-vertical (because $X_{\mathcal{L}}$ is a SODE and $j^1\tilde{Y}$ is a canonical lifting [5]) and $\Theta_{\mathcal{L}}$ is a π_1-semibasic form, so $i([j^1\tilde{Y}, X_{\mathcal{L}}])\Theta_{\mathcal{L}} = 0$. $\square$

As an immediate corollary of this theorem, we have the following result which recovers Noether's theorem.

Corollary 1. Let $\tilde{Y} \in \mathcal{X}(Q \times \mathbb{R})$ be a vector field such that $\tilde{Y}(t) = 1$ and ∇ the associated connection, $\nabla = \mathrm{d}t \otimes \tilde{Y}$. If the Lagrangian $\mathcal{L}$ is invariant by the prolongation of $\tilde{Y}$, that is $j^1\tilde{Y}(\mathcal{L}) = 0$, then the associated energy $E_{\mathcal{L}}^{\nabla}$ is invariant along the dynamical trajectories.

Furthermore, there is a relevant relationship between the first integrals of the dynamical vector field and the Lagrangian energy functions which is given in the next statement:

Proposition 4. If $\mathcal{L}$ is a Lagrangian function such that its associated Legendre transformation (see next section) is different from zero at every point, then every first integral of the dynamical vector field $X_{\mathcal{L}}$ is the energy associated to some connection.

Proof. Let ∇ be a connection such that its associated vector field Y verifies $(j^1\tilde{Y})\mathcal{L} = 0$, then it follows that $E_{\mathcal{L}}^{\nabla}$ is a conserved quantity. Conversely, if $f \in C^{\infty}(TQ \times \mathbb{R})$ verifies $X_{\mathcal{L}}(f) = 0$, take the connection associated to any vector field Y such that $i(j^1\tilde{Y})\Theta_{\mathcal{L}} = -f$. Taking into account that $\Theta_{\mathcal{L}}$ is π_1-semibasic and using the definition of the Legendre transformation $F\mathcal{L}$, this equation can be written as

$$Y \circ F\mathcal{L} + i\left(\frac{\partial}{\partial t}\right)\Theta_{\mathcal{L}} = -f$$

which obviously can be solved for Y if $F\mathcal{L}$ is different from zero at every point. Then we have that $E_{\mathcal{L}}^{\nabla} = f$. $\square$

This proposition proves that, assuming the hypothesis, every Noether invariant is the energy associated with some connection.

Examples.

(i) If $\mathcal{L}$ is an autonomous Lagrangian, it is invariant by the vector field $\tilde{Y}_0$ associated to the standard connection, then $E_{\mathcal{L}}^0$ is invariant.

(ii) Now, suppose that $\mathcal{L}$ is an autonomous Lagrangian with

$$\frac{\partial \mathcal{L}}{\partial q^{\mu}} = 0$$

for some q^μ. Taking the connection given by

$$\nabla = dt \otimes \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial q^\mu} \right)$$

the associated energy is

$$E_\mathcal{L}^\nabla = E_\mathcal{L}^0 - \frac{\partial \mathcal{L}}{\partial q^\mu}$$

and hence the corresponding components of the 'linear momentum' are invariant by the dynamics.

(iii) In the same way, if $\mathcal{L}$ is an autonomous Lagrangian and it is invariant by the prolongation of the vector field

$$\frac{\partial}{\partial t} + q^\mu \frac{\partial}{\partial q^\nu} - q^\nu \frac{\partial}{\partial q^\mu}$$

then the components of the 'angular momentum' are constants of the motion.

4. Hamiltonian formalism with non-standard connections

In the above section we have obtained that the Lagrangian vector field $X_\mathcal{L}$ does not depend on the connection and only the energy function must be redefined in this situation.

Next we are going to consider the Hamiltonian formulation of the problem associated with the Lagrangian $\mathcal{L}$ and the connection ∇. In order to achieve it, we need a *Hamiltonian function* and a *Hamilton–Cartan form*.

As usual, the geometric elements are defined in the bundles $T^*Q \times \mathbb{R} \xrightarrow{\tau_1} Q \times \mathbb{R} \xrightarrow{\pi} \mathbb{R}$. In addition, we have the *Legendre transformation* associated to $\mathcal{L}$, $F\mathcal{L}: TQ \times \mathbb{R} \to T^*Q \times \mathbb{R}$, defined as $F\mathcal{L}(x, t) := (F\mathcal{L}_t(x), t)$, for every $(x, t) \in TQ \times \mathbb{R}$, where $F\mathcal{L}_t: TQ \to T^*Q$ is the usual fibre derivative of the restriction of $\mathcal{L}$ obtained by considering its value for every fixed t. If (q^μ, t) is a local system in $Q \times \mathbb{R}$ and (q^μ, v^μ, t), (q^μ, p_μ, t) are the associated systems in $TQ \times \mathbb{R}$ and $T^*Q \times \mathbb{R}$, we have

$$F\mathcal{L}^* q^\mu = q^\mu \qquad F\mathcal{L}^* p_\mu = \frac{\partial \mathcal{L}}{\partial v^\mu} \qquad F\mathcal{L}^* t = t.$$

Then the Lagrangian function $\mathcal{L}$ is said to be *regular* iff $F\mathcal{L}$ is a local diffeomorphism and it is called *hyperregular* iff $F\mathcal{L}$ is a global diffeomorphism.

In order to construct the Hamilton–Cartan form, we need a 1-form in $T^*Q \times \mathbb{R}$. On the one hand we have the canonical forms in T^*Q, $\theta \in \Omega^1(T^*Q)$ and $\omega := -d\theta \in \Omega^2(T^*Q)$, and their pull-back to $T^*Q \times \mathbb{R}$ by means of the projection $\tau_{T^*Q}: T^*Q \times \mathbb{R} \to T^*Q$, so obtaining the forms $\theta_0 = \tau_{T^*Q}^* \theta$, $\omega_0 = \tau_{T^*Q}^* \omega$.

On the other hand, every connection ∇ allows the construction of another 1-form in $T^*Q \times \mathbb{R}$ in the following way:

Definition 4.   Consider $(\alpha, s) \in T^*Q \times \mathbb{R}$ and $u \in T_{(\alpha,s)}(T^*Q \times \mathbb{R})$, then $\theta_\nabla \in \Omega^1(T^*Q \times \mathbb{R})$ is defined by

$$\theta_\nabla((\alpha, s); u) := \alpha(v_\nabla(T\tau_1(u))).$$

(i) The form θ_∇ is called the *Liouville 1-form* associated with the connection ∇.
(ii) The form $\omega_\nabla := -d\theta_\nabla$ is called the *Liouville 2-form* associated with the connection ∇.

These forms are differentiable as can be deduced from their local expressions which, if

$$\tilde{\nabla} = dt \otimes \left(\frac{\partial}{\partial t} + \gamma^\mu \frac{\partial}{\partial q^\mu} \right)$$

and taking into account the expression of the vertical projection operator, are

$$\theta_\nabla = p_\mu dq^\mu - \gamma^\mu p_\mu dt \qquad \omega_\nabla = dq^\mu \wedge dp_\mu + \gamma^\mu dp_\mu \wedge dt + p_\mu \frac{\partial \gamma^\mu}{\partial q^\nu} dq^\nu \wedge dt.$$

Consider now the difference $\theta_0 - \theta_\nabla \equiv \eta$ which is a 1-form in $T^*Q \times \mathbb{R}$ whose local expression is $\eta = p_\mu \gamma^\mu dt$. Taking into account that θ_0 is a semibasic form, it can immediately be obtained that:

Proposition 5. If ∇ is a connection in $\pi \colon Q \times \mathbb{R} \to \mathbb{R}$ and $Y \in \mathcal{X}(Q \times \mathbb{R})$ its associated vector field, then

$$\eta = \theta_0(\mathcal{Y})dt$$

where $\mathcal{Y} \in \mathcal{X}(T^*Q \times \mathbb{R})$ is any extension of $\tilde{Y}$ from $Q \times \mathbb{R}$ to $T^*Q \times \mathbb{R}$. (We can take $\mathcal{Y} = j^1 \tilde{Y}$).

The other geometric element is the *Hamiltonian function*. In the standard situation, assuming that the Lagrangian function is hyperregular, this function h is defined by the equation $E_{\mathcal{L}}^0 = F\mathcal{L}^*h$. In the general case we have:

Definition 5. The *Hamiltonian function* associated with the Lagrangian $\mathcal{L}$ and the connection ∇ is the function $h^\nabla \in C^\infty(T^*Q \times \mathbb{R})$ such that

$$E_{\mathcal{L}}^\nabla = F\mathcal{L}^*h^\nabla.$$

As in the standard case, the existence and uniticity of this function is assured (at least locally) if we assume that $\mathcal{L}$ is a hyperregular (or regular) Lagrangian. In other cases, under certain hypotheses on $F\mathcal{L}$, it is possible to prove that $E_{\mathcal{L}}^\nabla$ and $E_{\mathcal{L}}^0$ are also $F\mathcal{L}$-projectable, so we can define h and h^∇.

A simple calculation in a local chart of natural coordinates leads to the following result:

Proposition 6.

$$h^\nabla = h - \eta \left(\frac{\partial}{\partial t} \right).$$

And, finally, we have the relations:

Proposition 7.

(i) $\qquad F\mathcal{L}^*\theta_0 = \Theta_{\mathcal{L}} + E_{\mathcal{L}}^0 dt.$

(ii) $\qquad F\mathcal{L}^*\theta_\nabla = \Theta_{\mathcal{L}} + E_{\mathcal{L}}^\nabla dt.$

Proof. The first one is a classical equality. The second one is proved directly in a local chart of natural coordinates. $\qquad \square$

Now we define:

Definition 6.   (i) The *standard Hamilton–Cartan (1 and 2)-forms* associated with the Hamiltonian function h are

$$\Theta_0 := \theta_0 - \text{h}dt \qquad \Omega_0 := -d\Theta_0 = \omega_0 + \text{dh} \wedge dt.$$

(ii) The *Hamilton–Cartan (1 and 2)-forms* associated with the connection ∇ and the Hamiltonian function h^∇ are

$$\Theta_\nabla := \theta_\nabla - \text{h}^\nabla dt \qquad \Omega_\nabla := -d\Theta_\nabla = \omega_\nabla + \text{dh}^\nabla \wedge dt.$$

Then we have the following result:

Proposition 8.   The Hamilton–Cartan forms associated with the standard and non-standard connections are the same:

$$\Theta_0 = \Theta_\nabla \qquad \Omega_0 = \Omega_\nabla.$$

Proof.   Both results arise taking into account that $\theta_0 - \theta_\nabla = \eta$ and the proposition (6).   □

Bearing in mind that the dynamical vector fields associated with the Hamiltonian functions h and h^∇ are respectively the solutions of the *Hamilton equations*

$$i(X_0)\Omega_0 = 0 \qquad i(X_0)dt = 1$$
$$i(X_\nabla)\Omega_\nabla = 0 \qquad i(X_\nabla)dt = 1$$

we conclude that these dynamical vector fields are the same: $X_\nabla = X_0$.

Hence, the connection ∇ deforms the geometric elements, but not the dynamical situation or, in other words, we can redefine the Hamiltonian function and the Liouville form in the Hamiltonian formalism in order to maintain the same dynamics.

As a final remark, one can easily check that $\Theta_\mathcal{L} = F\mathcal{L}^*\Theta_\nabla$. Therefore $\Omega_\mathcal{L} = F\mathcal{L}^*\Omega_\nabla$ and $X_\nabla = F\mathcal{L}_*X_\mathcal{L}$ for regular Lagrangians.

5. On the characterization of the energy by means of variational principles

We have defined above the Lagrangian energy function, justifying it geometrically. However, it can be considered as an *ad hoc* definition. Next we give another characterization of the energy which is based on variational principles and justifies the definition we have given.

First of all we need the following lemma:

Lemma 1.   Let $\beta \in \Omega^1(TQ \times \mathbb{R})$ and $f \in C^\infty(TQ \times \mathbb{R})$. The following conditions are equivalent:

(i) $\bar{\sigma}^*(f dt) = \bar{\sigma}^*\beta$ for every differentiable curve $\sigma: [a, b] \subset \mathbb{R} \to Q$.

(ii) $\int_{\bar{\sigma}} f dt = \int_{\bar{\sigma}} \beta$ for every differentiable curve $\sigma: [a, b] \subset \mathbb{R} \to Q$.

(where $\bar{\sigma}: [a, b] \subset \mathbb{R} \to TQ \times \mathbb{R}$ denotes the canonical lifting of σ).

Proof.   Trivially (i) $\Rightarrow$ (ii).

Conversely, if we suppose (i) is not true, then there exists one curve $\sigma: [a, b] \subset \mathbb{R} \to Q$ with $\bar{\sigma}^*(f dt - \beta) \neq 0$ and hence there is $s \in [a, b]$ and a closed neighbourhood V of s in $[a, b]$ such that, taking $\gamma: V \to Q$ with $\gamma = \sigma|_V$, then

$$\int_{\bar{\gamma}} (f dt - \beta) \neq 0$$

so (ii) is false. □

Next we state:

Proposition 9. The Lagrangian energy function introduced in definition 3 is the unique function in $TQ \times \mathbb{R}$ verifying the condition

$$\bar{\sigma}^*(E_{\mathcal{L}}^{\nabla}dt) = \bar{\sigma}^*(F\mathcal{L}^*\theta_{\nabla} - \mathcal{L}dt)$$

for every curve $\sigma : [a, b] \subset \mathbb{R} \to Q$.

Proof. Uniqueness: Let f and g be two functions verifying this condition. Obviously $\bar{\sigma}^*((f - g)dt) = 0$, but $0 = \bar{\sigma}^*((f - g)dt) = (f - g)(\bar{\sigma}(t))dt$, for every $t \in [a, b]$. Hence, $(f - g)(\bar{\sigma}(t)) = 0$, and this implies $f - g = 0$, because every point in $TQ \times \mathbb{R}$ is in the image of some curve $\bar{\sigma}$.

Existence. From proposition 7 we obtain

$$\bar{\sigma}^*(F\mathcal{L}^*\theta_{\nabla} - \mathcal{L}dt) = \bar{\sigma}^*(\Theta_{\mathcal{L}} + E_{\mathcal{L}}^{\nabla}dt + \mathcal{L}dt)$$
$$= \bar{\sigma}^*(d\mathcal{L} \circ \mathcal{V} - \mathcal{L}dt + E_{\mathcal{L}}^{\nabla}dt + \mathcal{L}dt) = \bar{\sigma}^*(E_{\mathcal{L}}^{\nabla}dt)$$

since $\bar{\sigma}^*(d\mathcal{L} \circ \mathcal{V}) = 0$. So, the energy function introduced in definition 3 satisfies this condition. □

From the lemma 1 we obtain, for every curve $\sigma : [a, b] \to Q$,

$$\int_{\bar{\sigma}} E_{\mathcal{L}}^{\nabla}dt = \int_{\bar{\sigma}} (F\mathcal{L}^*\theta_{\nabla} - \mathcal{L}dt)$$

therefore

$$\int_{\bar{\sigma}} \mathcal{L}dt = \int_{\bar{\sigma}} (F\mathcal{L}^*\theta_{\nabla} - E_{\mathcal{L}}^{\nabla}dt) = \int_{\bar{\sigma}} F\mathcal{L}^*(\theta_{\nabla} - h^{\nabla}dt)$$
$$= \int_{\bar{\sigma}} F\mathcal{L}^*(\theta_0 - hdt) = \int_{\bar{\sigma}} F\mathcal{L}^*\Theta_0 = \int_{F\mathcal{L}\circ\bar{\sigma}} \Theta_0$$

and this equality shows the equivalence between the *Hamilton principle of minimal action* (of the Lagrangian formalism) and the *Hamilton–Jacobi principle* (of the Hamiltonian formalism). Therefore, taking into account proposition 9, we have to conclude that the energy (as defined in definition 3) is the only function that realizes the equivalence between both variational principles. This fact justifies the definition given above.

6. Conclusions

The geometric description of non-autonomous mechanics is usually given using the natural connection induced by the trivial bundle structure of the phase spaces $TQ \times \mathbb{R}$ (for the Lagrangian formalism) and $T^*Q \times \mathbb{R}$ (for the Hamiltonian formalism). We have reformulated both formalisms starting from the choice of an arbitrary connection. As a consequence of this analysis, we have shown that the geometric construction of some elements of the theory depends on this choice; namely, the Lagrangian energy function of the Lagrangian formalism and the Hamiltonian formalism itself; in particular its geometrical structures (Liouville forms) and the Hamiltonian function. This fact is hidden in the usual geometric

descriptions because the connection used is the natural one associated to the trivial bundle structures.

We have generalized the geometric definition of the Lagrangian energy function, and consequently the Hamiltonian function, in order to take into account the use of non-standard connections. This definition characterizes the Lagrangian energy as the only function that realizes the equivalence between the Hamilton variational principle and the Hamilton–Jacobi principle.

The next step was to investigate the dynamical relevance of the choice of different connections. We have proved that the dynamics is insensitive to this choice because the Poincaré–Cartan forms of the Lagrangian formalism do not depend on the chosen connection, and the Hamilton–Cartan forms of the Hamiltonian one can be redefined in order to maintain the same dynamics.

It is worth pointing out that, from the physical point of view, the connection-dependence of the energy function can be understood as follows: for time-dependent systems, to take a non-standard connection can be interpreted as a change of the time action on $Q \times \mathbb{R}$ (which arises from taking $H(\nabla)$ instead of $\pi^* T\mathbb{R}$ as the horizontal sub-bundle). Then, it is reasonable that the energy function changes in its turn, since it can be considered as the conjugate function of 'time' (as is made evident in some geometrical descriptions of non-autonomous systems [7]). A significant result is the relationship between the dynamical variation of the connection-dependent energy and the variation of the Lagrangian with respect to the time-action induced by the connection.

A further consequence is that, for time-dependent dynamical systems described by a given Lagrangian function, we have proved that any first integral of the dynamics is the energy function for a suitable connection. Moreover, we have a means of obtaining different Lagrangian energy (or Hamiltonian) functions which are dynamically equivalent. This consists of taking different connections in order to construct these functions.

As a final remark, we trust that these results will help to clarify some aspects of the geometrical description of classical field theories.

Acknowledgments

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